Regularity of Solutions to the Stokes Equations under a Certain Nonlinear Boundary Condition

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Abstract

The regularity of a solution to the variational inequality for the Stokes equation is considered. The inequality describes the steady motion of the viscous incompressible fluid under a certain unilateral constrain of friction type. Firstly the solution is approximated by solutions to a regularized problem which is introduced by Yosida's regularization for a multi-valued operator. Then we establish a regularity result to the regularized problem. The regularity of the solution to the original inequality follows by the limiting argument.

1 Introduction

The present paper is concerned with the regularity of a solution to the following problem: Find $u \in K^1_\sigma(\Omega)$ and $p \in L^2(\Omega)$ satisfying

$$a(u, v-u) - (p, \mathrm{div}\,(v-u)) + j(v) - j(u) \geq (f, v-u), \quad (\forall v \in K^1(\Omega)). \quad (1.1)$$

Here and hereafter the following notation is employed: Ω is a bounded domain in $\mathbb{R}^m$, $m = 2$ or 3. The boundary $\partial\Omega$ is composed of two connected components Γ_0 and Γ. For the sake of simplicity, we assume that Γ_0 and Γ of class C^2 and that $\overline{\Gamma}_0 \cap \overline{\Gamma} = \emptyset$. The additional smoothness assumption on Γ will be specified later. We introduce

$$K^1(\Omega) = \{v \in H^1(\Omega)^m \mid v = 0 \text{ on } \Gamma_0\},$$

then $K^1_\sigma(\Omega)$ denotes the solenoidal subspace of $K^1(\Omega)$. $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $L^2(\Omega)^m$ according as scalar-valued functions or vector-valued functions. We set

$$a(u,v) = \frac{1}{2}\int_\Omega \sum_{1\le i,j\le m} e_{i,j}(u)e_{i,j}(v)\,dx, \quad e_{i,j}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

for $u = (u_1, \cdots, u_m)$ and $v = (v_1, \cdots, v_m)$. Finally

$$j(v) = \int_\Gamma g|v|\,ds, \quad (ds = \text{ the surface element of } \Gamma), \tag{1.2}$$

where g is a given scalar function defined on Γ.

As was described in Fujita and Kawarada [7], the variational inequality (1.1) arises in the study of the steady motions of the viscous incompressible fluid under the *frictional boundary condition*, where u denotes the flow velocity, p the pressure and f the external forces acting on the fluid, and g is called the modulus function of friction. We now review the boundary condition of this type. Let $\sigma(u,p)$ be the stress vector to Γ. That is, we let $\sigma(u,p) = S(u,p)n$, where $S(u,p) = [-p\delta_{i,j} + e_{i,j}(u)]$ stands for the stress tensor and n the unit outer normal to Γ. Then we pose on $\sigma(u,p)$ that

$$|\sigma(u,p)| \le g \tag{1.3}$$

and

$$\begin{cases} |\sigma(u,p)| < g \implies u = 0, \\ |\sigma(u,p)| = g \implies \begin{cases} u = 0 \text{ or } u \ne 0, \\ u \ne 0 \implies \sigma(u,p) = -gu/|u| \end{cases} \end{cases} \tag{1.4}$$

almost everywhere on Γ. The classical form of the frictional boundary value problem for the Stokes equations dealt with in [7] consists of

$$-\Delta u + \nabla p = f \text{ in } \Omega, \quad \operatorname{div} u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_0 \tag{1.5}$$

together with (1.3) and (1.4). The inequality (1.1) is a weak form of this problem.

Remark 1.1. To be more precise, j should be understood as the functional on $H^{1/2}(\Gamma)^m$;

$$j(\eta) = \int_\Gamma g|\eta|\,ds, \qquad (\eta \in H^{1/2}(\Gamma)^m).$$

However, for the sake of simplicity, we will regard j as the functional on $H^1(\Omega)$ through

$$j(v|_\Gamma) = \int_\Gamma g|v|_\Gamma|\,ds$$

and write it as (1.2).

The existence theorem was established in [7]. Assume that

$$f \in L^2(\Omega)^m, \quad g \in L^\infty(\Gamma), \quad g > 0 \text{ a.e. } \Gamma. \tag{1.6}$$

Then (1.1) admits of a solution $\{u,p\}$. The velocity part u is unique, and the uniqueness of the pressure part p depends on cases. We shall give an example about this issue in §5.

Henceforth we write $\|\cdot\|$, $\|\cdot\|_s$ and $\|\cdot\|_{s,\Gamma}$ instead of $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{H^s(\Gamma)}$, respectively. The trace γv on Γ of $v \in H^1(\Omega)$ is denoted by $v|_\Gamma$, where γ stands for the trace operator from $H^1(\Omega)$ into $H^{1/2}(\Gamma)$. The meaning of $v|_{\Gamma_0}$ and $v|_{\partial\Omega}$ is similar. For vector-valued functions, as long as there is no possibility of confusion, we shall use the same symbols. C denotes various generic constant. If it depends on parameters $q_1, \cdots, q_M$ which may not be numbers, we shall indicate it by $C = C(q_1, \cdots, q_M)$. Furthermore $\partial|\cdot|$ denotes the subdifferential of the function $|z| = (z_1^2 + \cdots + z_m^2)^{1/2}$.

The main purpose of this paper is to prove the following regularity result.

Theorem 1.1. *Let Γ be of class C^3. Assume that (1.6) and moreover that $g \in H^1(\Gamma)$. Let $\{u,p\}$ be a solution of (1.1). In particular, p is any corresponding pressure of u. Then $u \in H^2(\Omega)^m$ and $p \in H^1(\Omega)$ with*

$$\|u\|_2 + \|p\|_1 \leq C(\|f\| + \|g\|_{1,\Gamma}),$$

where $C = C(\Omega)$. Moreover we have $\sigma(u,p) \in H^{1/2}(\Gamma)^m$ and

$$-\sigma(u,p) \in g\partial|u| \quad \text{a. e. } \Gamma.$$

In order to prove Theorem 1.1, we follow the method of Brézis [2]. Namely, we approximate a solution $\{u,p\}$ of the *inequality* (1.1) by solutions $\{u_\varepsilon, p_\varepsilon\}$ of *equations* which are obtained by replacing j by a regular functional j_ε in (1.1). Then the regularity of $\{u_\varepsilon, p_\varepsilon\}$ is studied. Actually, for $\varepsilon > 0$, we introduce

$$j_\varepsilon(v) = \int_\Gamma g\rho_\varepsilon(v)\, ds, \qquad (v \in H^1(\Omega)^m),$$

where

$$\rho_\varepsilon(v) = \begin{cases} |v| - \varepsilon/2 & (|v| > \varepsilon), \\ |v|^2/(2\varepsilon) & (|v| \leq \varepsilon). \end{cases}$$

Then an approximate problem for (1.1) we shall consider is: Find $u_\varepsilon \in K^1_\sigma(\Omega)$ and $p_\varepsilon \in L^2(\Omega)$ satisfying

$$a(u_\varepsilon, v - u_\varepsilon) - (p_\varepsilon, \operatorname{div}(v - u_\varepsilon)) + j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) \geq (f, v - u_\varepsilon), \quad (\forall v \in K^1(\Omega)). \tag{1.7}$$

Concerning the approximate problem, we have the following three theorems.

Theorem 1.2. *Assume that (1.6) and let $\varepsilon > 0$. Then (1.7) admits a unique solution $\{u_\varepsilon, p_\varepsilon\}$ with*

$$\|u_\varepsilon\|_1 + \|p_\varepsilon\| \leq C(\|f\| + \|g\|_{L^2(\Gamma)}),$$

where $C = C(\Omega)$. *Furthermore,* $\{u_\varepsilon, p_\varepsilon\}$ *is a weak solution of (1.5) together with*

$$-\sigma(u_\varepsilon, p_\varepsilon) = g\alpha_\varepsilon(u_\varepsilon) \quad \text{a. e. } \Gamma. \quad (\textit{In particular } \sigma(u_\varepsilon, p_\varepsilon) \in L^2(\Gamma)^m), \tag{1.8}$$

where we have put

$$\alpha_\varepsilon(v) = \begin{cases} v/|v| & (|v| > \varepsilon) \\ v/\varepsilon & (|v| \le \varepsilon). \end{cases}$$

Remark 1.2. In Theorem 1.2, $\sigma(u,p)$ is understood as a functional on $H^{1/2}(\Gamma)^m$ defined by

$$\langle \sigma, \eta \rangle = a(u_\varepsilon, \psi_\eta) - (p_\varepsilon, \operatorname{div} \psi_\eta) - (f, \psi_\eta), \quad (\forall \eta \in H^{1/2}(\Gamma)^m),$$

where $\psi_\eta \in K^1(\Omega)$ is any extension of η.

Remark 1.3. Our choice of the regularized functional is based on *Yosida's regularization.* Namely, Yosida's regularization of $\partial|\cdot|$ coincides with $\alpha_\varepsilon = \nabla \rho_\varepsilon$.

Theorem 1.3. *Assume that (1.6) and let* $\varepsilon > 0$. *Let* $\{u, p\}$ *and* $\{u_\varepsilon, p_\varepsilon\}$ *be solutions of (1.1) and (1.7), respectively. Then we have:*

$$\|u_\varepsilon - u\|_1 + \|\tilde{p}_\varepsilon - \tilde{p}\| \le C(\Omega, g)\sqrt{\varepsilon}, \tag{1.9}$$

where $\tilde{p} = p - |\Omega|^{-1}(p, 1)$, $\tilde{p_\varepsilon} = p_\varepsilon - |\Omega|^{-1}(p_\varepsilon, 1)$ *and* $|\Omega|$ *indicates the measure of* Ω *in* $\mathbb{R}^m$

Theorem 1.4. *Let* Γ *be of class* C^3 *and let* $\varepsilon > 0$. *Assume that (1.6) and moreover that* $g \in H^1(\Gamma)$. *Let* $\{u_\varepsilon, p_\varepsilon\}$ *be a solution of (1.7). Then* $u_\varepsilon \in H^2(\Omega)^m$ *and* $p_\varepsilon \in H^1(\Omega)$ *with*

$$\|u_\varepsilon\|_2 + \|p_\varepsilon\|_1 \le C(\|f\| + \|g\|_{1,\Gamma}). \tag{1.10}$$

It should be kept in mind that $C = C(\Omega)$ *does not depend on* ε.

Now we can state:

Proof of Theorem 1.1. Let $\varepsilon > 0$, and let $\{u_\varepsilon, p_\varepsilon\}$ be a solution of (1.7). By virtue of Theorem 1.4, sequences $\|u_\varepsilon\|_2$ and $\|p_\varepsilon\|_1$ are bounded as $\varepsilon \downarrow 0$, respectively. Hence, there are subsequences $\{u_{\varepsilon'}\}$ and $\{p_{\varepsilon'}\}$ such that

$$u_{\varepsilon'} \to u^* \text{ weakly in } H^2(\Omega)^m, \quad p_{\varepsilon'} \to p^* \text{ weakly in } H^1(\Omega)$$

and

$$\|u^*\|_2 + \|p^*\|_1 \le C(\|f\| + \|g\|_{1,\Gamma}).$$

According to Theorem 1.3, $\{u^*, p^*\}$ is a solution of (1.1). Next let $\{u, p\}$ be any solution of (1.1). By the uniqueness of the velocity part, we have $u = u^*$. On the other hand, $p - p^* = k$ and a constant k is restricted via (1.3). Therefore $p \in H^1(\Omega)$, and we deduce

$$\sigma(u, p) - \sigma(u, p^*) = kn \quad \text{a. e. } \Gamma.$$

This, together with (1.3), implies that $|k| \le 2g$ holds almost everywhere on Γ. Hence $|k| \le 2|\Gamma|^{-1/2}\|g\|_{L^2(\Gamma)}$, where $|\Gamma|$ denotes the measure of Γ in $\mathbb{R}^{m-1}$. By making use of this estimate, we have

$$\begin{aligned}\|u\|_2 + \|p\|_1 &\le \|u\|_2 + \|p^*\|_1 + |k|\sqrt{|\Omega|}\\ &\le C(\|f\| + \|g\|_{1,\Gamma}),\end{aligned}$$

which completes the proof. □

The rest of the present paper is composed of the following sections:

§2. Proof of Theorem 1.2
§3. Proof of Theorem 1.3
§4. Proof of Theorem 1.4
§5. Remarks

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2 Proof of Theorem 1.2

As a preliminary to prove Theorem 1.2, we describe a decomposition theorem concerning $K^1(\Omega)$ which is essentially due to Solonnikov and Ščadilov [14].

Through Riesz's representation theorem, we define an operator B from $L^2(\Omega)$ to $K^1(\Omega)$ by

$$(Bq, v)_{H^1(\Omega)^m} = (q, \operatorname{div} v), \quad (\forall q \in L^2(\Omega);\ \forall v \in K^1(\Omega)). \tag{2.1}$$

Lemma 2.1. *The range $R(B)$ of B forms a closed subspace of $K^1(\Omega)$. Moreover we have the orthogonal decomposition*

$$K^1(\Omega) = R(B) \oplus K^1_\sigma(\Omega).$$

Proof. By taking $v = Bq$ in (2.1), we have $\|Bq\|_1 \le \|q\|$; Thus, the linear operator B is bounded. On the other hand, as will be verified later, for any $q \in L^2(\Omega)$, we can take $\tilde{v} \in K^1(\Omega)$ satisfying

$$\operatorname{div} \tilde{v} = q \text{ in } \Omega, \quad \|\tilde{v}\|_1 \le C'\|q\|.$$

Substituting $v = \tilde{v}$ into (2.1), we obtain $\|q\| \le C'\|Bq\|_1$, which means that the inverse operator B^{-1} is also bounded. Consequently, we have

$$C\|q\| \le \|Bq\|_1 \le C'\|q\| \quad (\forall q \in L^2(\Omega)). \tag{2.2}$$

Therefore the closedness of $R(B)$ follows. To prove $R(B)^\perp = K^1_\sigma(\Omega)$ is an easy task, where $R(B)^\perp$ denotes the orthogonal complement of $R(B)$ in $K^1(\Omega)$. It remains to verify the existence of $\tilde{v}$ appeared above. We take $w \in H^2(\Omega)$ subject to $\Delta w = p$ in Ω and introduce $\zeta \in H^{1/2}(\partial\Omega)^m$ by

$$\zeta = \begin{cases} \nabla w & \text{on } \Gamma_0 \\ \lambda n & \text{on } \Gamma, \end{cases}$$

where a constant λ is chosen as $\lambda = |\Gamma|^{-1}(\nabla w, n)_{L^2(\Gamma_0)}$. Then, since $(\zeta, n)_{L^2(\partial\Omega)} = 0$, there is a $v_0 \in H^1(\Omega)^m$ such that div $v_0 = 0$ in Ω and $\|v_0\|_1 \le C\|\zeta\|_{1/2,\partial\Omega} \le C\|p\|$. The desired function is $\tilde{v} = \nabla w - v_0$. □

Proof of Theorem 1.2. Since j_ε is a convex, lower semi-continuous proper functional, from standard theory of convex analysis (for example, Glowinski [8]), the minimization problem: Find $u \in K^1_\sigma(\Omega)$ satisfying

$$\mathcal{J}_\varepsilon(u) = \inf_{v \in K^1_\sigma(\Omega)} \mathcal{J}_\varepsilon(v), \quad \mathcal{J}_\varepsilon(v) = \frac{1}{2}a(v,v) - (f,v) + j_\varepsilon(v)$$

has a unique solution u which is characterized by

$$a(u, v-u) + j_\varepsilon(v) - j_\varepsilon(u) \ge (f, v-u), \quad (\forall v \in K^1_\sigma(\Omega)). \tag{2.3}$$

Following Solonnikov and Ščadilov [14], we shall prove that a scalar function p can be taken as $\{u, p\}$ solves (1.7). Substituting into (2.3) $v = u \pm t\phi$ with arbitrary $\phi \in K^1_\sigma(\Omega)$, $t > 0$ and letting $t \to 0$, we obtain

$$a(u,\phi) + \int_\Gamma g\alpha_\varepsilon(u) \cdot \phi \, ds = (f,\phi), \quad (\forall \phi \in K^1_\sigma(\Omega)).$$

We introduce a linear functional in $K^1(\Omega)$ by setting

$$F(\psi) = a(u,\psi) + \int_\Gamma g\alpha_\varepsilon(u) \cdot \psi \, ds - (f,\psi) \quad (\forall \psi \in K^1(\Omega)).$$

Then by making use of Lemma 2.1 and Riesz's representation theorem, we can easily show that there exists a $p \in L^2(\Omega)$ satisfying

$$F(\psi) = (p, \operatorname{div} \psi) \quad (\forall \psi \in K^1(\Omega)).$$

This yields

$$a(u,\psi) - (p, \operatorname{div} \psi) + \int_\Gamma g\alpha_\varepsilon(u) \cdot \psi \, ds = (f,\psi) \quad (\forall \psi \in K^1(\Omega)). \tag{2.4}$$

The uniqueness of p is obvious on account of (2.2). Thanks to the convexity of j_ε,

$$\int_\Gamma g\alpha_\varepsilon(v) \cdot (w - v) \, ds \le j_\varepsilon(w) - j_\varepsilon(v), \quad (\forall v, w \in H^1(\Omega)^m).$$

By using this, we can easy verify that $\{u,p\}$ solves (1.7). On the other hand, (2.4) leads to

$$\langle \sigma, \psi \rangle + \int_\Gamma g\alpha_\varepsilon(u) \cdot \psi \, ds = 0 \quad (\forall \eta \in H^{1/2}(\Gamma)^m),$$

where $\psi \in K^1(\Omega)$ is any extension of η. Consequently, it follows from $g\alpha_\varepsilon(u) \in L^2(\Gamma)^m$ that $\sigma(u,p) \in L^2(\Gamma)^m$ and

$$-\sigma(u,p) = g\alpha_\varepsilon(u) \quad \text{a.e. } \Gamma,$$

which completes the proof. □

Remark 2.1. The bilinear form a is continuous in $H^1(\Omega)$; There is a constant $\delta_0 > 0$ depending only on Ω such that

$$a(u,v) \le \delta_0 \|u\|_1 \|v\|_1 \quad (\forall u, v \in H^1(\Omega)^m).$$

Moreover, a is coercive in $K^1(\Omega)$. In fact, Korn's inequality (e.g., for example, Duvaut and Lions [5]) implies that

$$a(v,v) \ge \delta_1 \|v\|_1^2 \quad (\forall v \in K^1(\Omega))$$

with a domain constant $\delta_1 > 0$.

3 Proof of Theorem 1.3

The derivation of

$$\|u - u_\varepsilon\|_1 \le C(\Omega, g)\sqrt{\varepsilon} \tag{3.1}$$

is same as the proof of Theorem 10.4 in Kikuchi and Oden [9]. In particular, we can take a constant as $C(\Omega, g) = (|\Gamma| \cdot \|g\|_{L^\infty(\Gamma)}/\delta_1)^{1/2}$. We proceed to the estimate of the pressure part; Namely, we shall prove

$$\|\tilde{p}_\varepsilon - \tilde{p}\| \le C'(\Omega, g)\sqrt{\varepsilon}. \tag{3.2}$$

Putting $q_\varepsilon = \tilde{p}_\varepsilon - \tilde{p}$, we have

$$a(u - u_\varepsilon, \phi) = (q_\varepsilon, \operatorname{div} \phi) \quad (\forall \phi \in H_0^1(\Omega)^m). \tag{3.3}$$

In view of Babušca-Aziz's lemma ([1]), we can take $w_\varepsilon \in H_0^1(\Omega)^m$ subject to div $w_\varepsilon = q_\varepsilon$ in Ω and $\|w_\varepsilon\|_1 \le C''(\Omega)\|q_\varepsilon\|$. Now substituting $\phi = w_\varepsilon$ into (3.3), we deduce

$$\|q_\varepsilon\|^2 = a(u - u_\varepsilon, w_\varepsilon) \le \delta_0 \|u - u_\varepsilon\|_1 \|w_\varepsilon\|_1 \le \delta_0 C''(\Omega)\|u - u_\varepsilon\|_1 \|q_\varepsilon\|.$$

Combining this with (3.1), we get (3.2) and therefore (1.9). □

4 Proof of Theorem 1.4

We firstly review the well-known regularity result for the Stokes equations under the Neumann boundary condition.

Lemma 4.1. *Let $f \in L^2(\Omega)^m$ and $\omega \in H^{1/2}(\Gamma)^m$. Suppose that $\{u,p\} \in H^1(\Omega)^m \times L^2(\Omega)$ is a weak solution of (1.5) with $\sigma(u,p) = \omega$ on Γ. (See Remark 1.2). Then $u \in H^2(\Omega)^m$ and $p \in H^1(\Omega)$ with*

$$\|u\|_2 + \|p\|_1 \leq C(\|f\| + \|\omega\|_{1/2,\Gamma}).$$

Lemma 4.1 in the case of $\omega \equiv 0$ was described in Solonnikov [13] with a mention on Solonnikov and Ščadilov [14] concerning the method of the proof. However it seems that the complete proof for the case of $\omega \not\equiv 0$ is not explicitly stated in these papers; In this connection, we refer to a forthcoming paper Saito [11].

Lemma 4.2. *Let $\{u,p\}$ be a solution of (1.7), and put $\omega_\varepsilon = g\alpha_\varepsilon(u_\varepsilon)$. Then we have $\omega_\varepsilon \in H^{1/2}(\Gamma)^m$. Thus, from Lemma 4.1, $u_\varepsilon \in H^2(\Omega)^m$ and $p_\varepsilon \in H^1(\Omega)$.*

Proof. Firstly we verify that $\alpha_\varepsilon(u_\varepsilon) \in H^{1/2}(\Gamma)^m$ with

$$\|\alpha_\varepsilon(u_\varepsilon)\|_{1/2,\Gamma} \leq C(\varepsilon,\Gamma)\|u_\varepsilon|_\Gamma\|_{1/2,\Gamma}. \tag{4.1}$$

This is essentially due to Brézis [2], where he dealt with the scalar case, and it is possible to extend his result into our vector-values case. In fact, we can prove that

1° α_ε is a bounded operator in $H^1(\Gamma)^m$.

2° $\|\alpha_\varepsilon(\omega)\|_{1,\Gamma} \leq \varepsilon^{-1}C(\Gamma)\|\omega\|_{1,\Gamma}$, $(\forall\omega \in H^1(\Gamma)^m)$.

3° $\|\alpha_\varepsilon(\omega) - \alpha_\varepsilon(\tilde\omega)\|_{L^2(\Gamma)^m} \leq \varepsilon^{-1}C(\Gamma)\|\omega - \tilde\omega\|_{L^2(\Gamma)^m}$, $(\forall\omega,\tilde\omega \in L^2(\Gamma)^m)$.

The assertions 2° and 3° are easy consequences of a property of Yosida's regularization. As a result, we can apply a nonlinear interpolation theorem by J.L. Lions (Theorem 3.1, [10]) and obtain that α_ε is an operator on $H^\theta(\Gamma)^m$ with

$$\|\alpha_\varepsilon(\omega)\|_{\theta,\Gamma} \leq \varepsilon^{-1}C(\Gamma)\|\omega\|_{\theta,\Gamma}, \quad (\forall\omega \in H^\theta(\Gamma)^m)$$

where $0 \leq \theta \leq 1$. Taking $\theta = 1/2$, we have

$$\|\alpha_\varepsilon(\omega)\|_{1/2,\Gamma} \leq \varepsilon^{-1}C(\Gamma)\|\omega\|_{1/2,\Gamma}, \quad (\forall\omega \in H^{1/2}(\Gamma)^m),$$

which implies (4.1).

Next let us denote by $\tilde g \in H^1(\Omega)$ the weak harmonic extension of $g \in H^{1/2}(\Gamma)$:

$$\Delta\tilde g = 0 \text{ in } \Omega, \quad \tilde g = 0 \text{ on } \Gamma_0, \quad \tilde g = g \text{ on } \Gamma.$$

It follows from the maximum principle that $\|\tilde g\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Gamma)}$. On the other hand, we take the weak harmonic extension $\tilde\alpha_\varepsilon \in H^1(\Omega)$ of $\alpha_\varepsilon(v)$. That is, we extend each component of $\alpha_\varepsilon(v)$ into Ω by the harmonic function. By the definition of α_ε and again the maximum principle, we have $\|\tilde\alpha_\varepsilon\|_{L^\infty(\Omega)} \leq \|\alpha_\varepsilon(v)\|_{L^\infty(\Gamma)} \leq m$. Since $\tilde g\tilde\alpha_\varepsilon \in H^1(\Omega)^m$, the trace $g\alpha_\varepsilon(v) \in H^{1/2}(\Gamma)^m$. □

In order to derive the estimate (1.10) which is independent of ε, we need another device.

Lemma 4.3. *Under the same assumptions of Theorem 1.4,*

$$\|\beta_\varepsilon\|_{3/2,\Gamma} \leq C(\Omega)(\|f\| + \|g\|_{1,\Gamma}), \tag{4.2}$$

where $\beta_\varepsilon = u_\varepsilon|_\Gamma$.

At this stage, we can state

Proof of Theorem 1.4. Let $\varepsilon > 0$. Lemma 4.3 implies that a solution $\{u_\varepsilon, p_\varepsilon\}$ of (1.7) satisfies the usual Dirichlet boundary value problem composed of (1.5) and

$$u_\varepsilon = \beta_\varepsilon \quad \text{on } \Gamma.$$

Therefore, by virtue of Cattabriga's regularity result ([3]), we have

$$\|u_\varepsilon\|_2 + \|p_\varepsilon\|_1 \leq C(\Omega)(\|f\| + \|\beta_\varepsilon\|_{3/2,\Gamma}).$$

By taking account of (4.2), we finally have (1.10). □

It remains to prove Lemma 4.3.

Proof of Lemma 4.3. Let $x_0 \in \Gamma$. Then there exist a neighborhood $U \subset \mathbb{R}^m_x$ of x_0 and a one-to-one mapping $\Phi = (\Phi_1, \cdots, \Phi_m)$ from U onto $\tilde{U} \subset \mathbb{R}^m_y$ enjoying the following properties (See, for example, §I-2 in Wolka [15]):

1° Φ is a C^3-diffeomorphism;

2° $\Phi(x_0) = 0$;

3° $\Phi(U \cap \Omega) = Q_R \equiv \{y = (y', y_m) \in \mathbb{R}^{m-1} \times \mathbb{R};\ |y'| < R,\ 0 < y_m < R\}$;

4° $\Phi(U \cap \Gamma) = \gamma_R \equiv \{y = (y', y_m) \in \mathbb{R}^{m-1} \times \mathbb{R};\ |y'| < R,\ y_m = 0\}$;

5° $\dfrac{\partial \Phi_m}{\partial x_j} = -n_j$ on $U \cap \Gamma$, where $n = (n_1, \cdots, n_m)$.

Setting $y = \Phi(x) = (\Phi_1(x) \cdots, \Phi_m(x))$ and introducing

$$v(y) = u(x), \qquad q(y) = p(x)$$

and

$$\tilde{f}(y) = f(x), \quad \tilde{g}(y) = g(x), \quad \tilde{\alpha}(v(y)) = \alpha(u(x)),$$

we have for $j = 1, \cdots, m$

$$-\sum_{1\leq k,l\leq m} \frac{\partial}{\partial y_l} E_{j,k,l} + \sum_{k=1}^{m} \frac{\partial \Phi_k}{\partial x_j}\frac{\partial q}{\partial y_k} = F_j \qquad \text{in } Q_R, \tag{4.3}$$

$$\tilde{\sigma}_j(v, q) = -\tilde{g}\tilde{\alpha}_j(v) \qquad \text{on } \gamma_R \tag{4.4}$$

and

$$\sum_{1\le k,l\le m} \frac{\partial \Phi_k}{\partial x_l}\frac{\partial v_l}{\partial y_k} = 0 \text{ in } Q_R. \tag{4.5}$$

Here

$$E_{j,k,l} = b_{kl}\frac{\partial v_j}{\partial y_k} + \frac{\partial \Phi_k}{\partial x_j}\sum_{\nu=1}^{m}\frac{\partial \Phi_l}{\partial x_\nu}\frac{\partial v_\nu}{\partial y_k}, \qquad b_{kl} = \sum_{\nu=1}^{m}\frac{\partial \Phi_k}{\partial x_\nu}\frac{\partial \Phi_l}{\partial x_\nu},$$

$$F_j = \tilde f_j - \sum_{\nu=1}^{m} d_\nu \frac{\partial v_j}{\partial y_\nu}, \qquad \tilde\sigma_j(v,q) = q\frac{\partial \Phi_m}{\partial x_j} - \sum_{\nu=1}^{m} E_{j,\nu,m}$$

and d_j denotes a bounded function in Q_R depending on Φ, $\nabla\Phi$ and $\nabla^2\Phi$. In fact, (4.3) follows from

$$-\sum_{l=1}^{m}\frac{\partial}{\partial x_l}e_{j,l}(u) + \frac{\partial p}{\partial x_j} = f_j,$$

which is an equivalent expression to $-\Delta u_j + \partial p/\partial x_j = f_j$ under div $u = 0$. On the other hand, div $u = 0$ and (1.8) imply (4.5) and (4.4), respectively.

Let φ_j be a function such that $\varphi_j(y', y_m) = 0$ for $|y'| \ge R$ or for $y_m \ge R$. Multiplying by φ_j both sides of (4.3) and integrating over Q_R, we have by integration by parts

$$\sum_{k,l}\int_{Q_R} E_{j,k,l}\frac{\partial \varphi_j}{\partial y_l} - \sum_{k}\int_{Q_R} q\frac{\partial}{\partial y_k}\Big(\frac{\partial \Phi_k}{\partial x_j}\varphi_j\Big) = \int_{\gamma_R}\tilde\sigma_j(v,q)\varphi_j\,dy' + \int_{Q_R}F_j\varphi_j.$$

Hence, writing $\varphi = (\varphi_1, \cdots, \varphi_m)$, we obtain

$$\sum_{k,l}\int_{Q_R}\Big[b_{k,l}\frac{\partial v}{\partial y_k}\frac{\partial \varphi}{\partial y_l} + \Big(\frac{\partial v}{\partial y_k}\nabla_x\Phi_l\Big)\Big(\frac{\partial \varphi}{\partial y_l}\nabla_x\Phi_k\Big)\Big]$$
$$- \sum_{k}\int_{Q_R} q\frac{\partial}{\partial y_k}\Big(\varphi\nabla_x\Phi_k\Big) = \int_{\gamma_R}\tilde\sigma(v,q)\varphi\,dy' + \int_{Q_R}F\varphi. \tag{4.6}$$

Now we choose φ_j as

$$\varphi_j = -\sum_{i=1}^{m-1}\frac{\partial}{\partial y_i}\Big(\eta^2\frac{\partial v_j}{\partial y_i}\Big),$$

where $\eta \in C^\infty(\mathbb{R}^m_y)$ stands for a cut-off function subject to

$$0 \le \eta \le 1 \text{ in } \mathbb{R}^m_y, \quad \eta = 1 \text{ in } Q_{R/2}, \quad \eta = 0 \text{ for } |y'| \ge R \text{ or } y_m \ge R.$$

If we write (4.6) as $I_1 - I_2 = I_3 + I_4$, then we deduce

$$\begin{aligned} I_1 \geq c_1 \sum_{i=1}^{m-1}\sum_{k=1}^{m}\Big\|\eta\frac{\partial^2 v}{\partial y_i \partial y_k}\Big\|^2 \\ - c_2\|v\|_{1,Q_R}\sum_{i=1}^{m-1}\sum_{k=1}^{m}\Big\|\eta\frac{\partial^2 v}{\partial y_i \partial y_k}\Big\| - c_3\|v\|^2_{1,Q_R}, \end{aligned} \tag{4.7}$$

$$|I_2| \leq c_4\|q\|_{Q_R}\Big(\|v\|_{1,Q_R} + \sum_{i=1}^{m-1}\sum_{k=1}^{m}\Big\|\eta\frac{\partial^2 v}{\partial y_i \partial y_k}\Big\|\Big), \tag{4.8}$$

$$I_3 \leq c_5\|\tilde{g}\|_{1,\gamma_R}\|v\|_{1,Q_R}, \tag{4.9}$$

and

$$I_4 \leq c_6(\|\tilde{f}\|_{Q_R} + \|v\|_{1,Q_R})\Big(\|v\|_{1,Q_R} + \sum_{i=1}^{m-1}\sum_{k=1}^{m}\Big\|\eta\frac{\partial^2 v}{\partial y_i \partial y_k}\Big\|\Big) \tag{4.10}$$

Here we check only (4.9). Other inequalities are verified by Saito [12]. Firstly, we note that

$$\nabla_s\tilde{\alpha}(s)t \cdot t \geq 0 \qquad \forall s, t \in \mathbb{R}^m.$$

Hence

$$\begin{aligned} I_3 &= -\sum_{i=1}^{m-1}\int_{\gamma_R}\Big\{\tilde{g}\eta^2\Big[(\nabla_v\tilde{\alpha}(v))\frac{\partial v}{\partial y_i}\cdot\frac{\partial v}{\partial y_i}\Big] + \Big(\frac{\partial \tilde{g}}{\partial y_i}\tilde{\alpha}(v)\Big)\cdot\eta^2\frac{\partial v}{\partial y_i}\Big\}\,dy' \\ &\leq \sum_{i=1}^{m-1}\Big\|\frac{\partial \tilde{g}}{\partial y_i}\Big\|_{\gamma_R}\Big\|\frac{\partial v}{\partial y_i}\Big\|_{\gamma_R} \leq c_7\|\tilde{g}\|_{1,\gamma_R}\|\nabla v\|_{Q_R}. \end{aligned}$$

Taking into account of (4.7)-(4.10), we have

$$\begin{aligned} \sum_{i=1}^{m-1}\sum_{k=1}^{m}\Big\|\eta\frac{\partial^2 v}{\partial y_i \partial y_k}\Big\|^2 \leq c_8(\|v\|_{1,Q_R} + \|q\|_{Q_R} + \|\tilde{g}\|_{1,\gamma_R} + \|\tilde{f}\|_{Q_R})\cdot \\ \Big(\|v\|_{1,Q_R} + \sum_{i=1}^{m-1}\sum_{k=1}^{m}\Big\|\eta\frac{\partial^2 v}{\partial y_i \partial y_k}\Big\|\Big). \end{aligned}$$

This yields

$$\sum_{i=1}^{m-1}\sum_{k=1}^{m}\Big\|\eta\frac{\partial^2 v}{\partial y_i \partial y_k}\Big\| \leq c_9(\|v\|_{1,Q_R} + \|q\|_{Q_R} + \|\tilde{g}\|_{1,\gamma_R} + \|\tilde{f}\|_{Q_R}),$$

and therefore v

$$\sum_{i=1}^{m-1}\Big\|\frac{\partial v}{\partial y_i}\Big\|_{1,Q_{R'}} \leq c_{10}(\|v\|_{1,Q_R} + \|q\|_{Q_R} + \|\tilde{g}\|_{1,\gamma_R} + \|\tilde{f}\|_{Q_R})$$

with $R' = R/2$. According to this inequality, introducing

$$\mu_j = \frac{\partial v}{\partial y_j}\Big|_{\gamma_{R'}}, \quad (j = 1, \cdots, m-1)$$

we arrive at

$$\|\mu_j\|_{1/2,\gamma_{R'}} \le c_{11}(\|v\|_{1,Q_R} + \|q\|_{Q_R} + \|\tilde{g}\|_{1,\gamma_R} + \|\tilde{f}\|_{Q_R}), \quad (j = 1, \cdots, m-1).$$

This means that all tangential derivatives of $v|_{\gamma_{R'}}$ belongs to $H^{1/2}(\gamma_{R'})$. Therefore, $v|_{\gamma_{R'}} \in H^{3/2}(\gamma_{R'})^m$ and

$$\|v|_{\gamma_{R'}}\|_{3/2,\gamma_{R'}} \le c_{12}(\|v\|_{1,Q_R} + \|q\|_{Q_R} + \|\tilde{g}\|_{1,\gamma_R} + \|\tilde{f}\|_{Q_R})$$

Summing up the above estimates, we finally have

$$\|\beta_\varepsilon\|_{3/2,\Gamma} \le c_{13}(\|u\|_{1,\Omega} + \|p\|_\Omega + \|g\|_{1,\Gamma} + \|f\|_\Omega).$$

Therefore we have established the desired (4.2). □

Remark 4.1. As is described in [12], the inequality (4.7) is a consequence of Korn's inequality, and (4.8) is that of (4.5). Furthermore, the constant c_4 depends on the third derivatives of Φ so that we need to assume that Γ is of class C^3.

5 Remarks

(A) Non-uniqueness of the pressure part. In general, a corresponding pressure p of the velocity u which is a solution of (1.5)(1.3)(1.4) is not unique. We give a simple example. For the time being, we employ the polar coordinates $x = (r, \theta)$ in $\mathbb{R}^2$. We assume that

$$\Omega = \{(r,\theta); 1 < r < \sqrt{2}\}, \quad \Gamma_0 = \{r = 1\}, \quad \Gamma = \{r = \sqrt{2}\},$$

and set $e_r = (\cos\theta, \sin\theta)$, $e_\theta = (-\sin\theta, \cos\theta)$. Put $u(r,\theta) = w(r)e_\theta$ and $p(r) = \kappa r$, where $w(r) = 2/r - r$ and $\kappa > 0$ is a constant. We notice that $\{u, p\}$ solves

$$-\Delta u + \nabla p = \kappa e_r, \quad \text{div } u = 0 \text{ in } \Omega, \quad u|_{\Gamma_0} = e_\theta, \quad u|_\Gamma = 0. \tag{5.1}$$

Moreover the stress vector is written as $\sigma(u,p) = -p(r)e_r + W(r)e_\theta$, where $W(r) = -4w(r)/r^2$. Hence $|\sigma(u,p)| = \sqrt{2\kappa^2 + 4}$.

Firstly we define $g = \sqrt{2\kappa^2 + 4} + 1$. Then $|\sigma(u,p)| < g$ and $u|_\Gamma = 0$ hold. Therefore $\{u, p\}$ is a solution of (5.1)(1.3)(1.4).

Next put $p_c = p + c$ with a constant c, and let $c_0 < c_1$ be roots of the equation $2\sqrt{2}\kappa c + c^2 = 1$. Then the following facts are hold true: (i) if $c_0 \le c \le c_1$ then $\{u, p_c\}$ is also a solution of (5.1)(1.3)(1.4). (ii) if $c < c_0$ or $c_1 < c$ then $\{u, p_c\}$ is not a solution. Namely, in this case, p is not unique and the non-uniqueness is restricted through (1.3).

On the other hand, if non-trivial movement ($u \ne 0$) takes place on a portion $\Gamma_1 \subset \Gamma$, then p is uniquely determined.

(B) Other flow problems. From the view point of physics, some modifications of (1.1) are much more interesting. Actually, H. Fujita [6] studied the Navier-Stokes or the Stokes equations under the *leak or slip boundary conditions of friction type.* The similar results about the regularities of solutions for these problems could be obtained by the same method presented in this paper. See, for more detail, Saito [12].

(C) Problems in elasticity theory. Our method is applicable for some problems in elasticity theory. For instance, we can give another proof of a regularity theorem described in M. Cocu and A.R. Radoslovescu [4], where they dealt with Signorini problem with non-local friction.

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