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On the domain-dependence of convergence rate in a domain decomposition method for the Stokes equations

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Abstract — A certain domain decomposition iterative algorithm for the Stokes equations is considered. The aim of the present paper is to study relationships between the convergence speed of iteration and the shapes of subdomains. Our consideration is restricted to the continuous problem and to the case in which the whole domain is divided into two non-overlapping subdomains. Then, introducing a certain geometric condition, we can derive explicit decay rates of the error on an artificial boundary. Also, a new important role of the so-called inf-sup constant is revealed. The proof is accomplished by means of an operator theory.

Keywords:. domain decomposition method, convergence rate, Stokes equations, inf-sup constant.

AMS(MOS) subject classification: 65N55, 76D07.

The present paper is concerned with some analytical foundations of the domain decomposition method (DDM). In particular our attention is focused on the *domain-dependence* of the convergence speed of some domain decomposition iterative scheme. This means that we shall make an analytical study on sensitive relations between the rate of convergence of iterations and the shape of decomposed domains. The problem of this sort was firstly considered by H. Fujita in 1995. Specifically, under some geometric conditions between subdomains, for example Condition (I) described in Section 2, he succeeded in deriving explicit decay rates of the error and obtaining some information on the optimal choice of relaxation parameter ([6, 7]). Several generalizations of Fujita's results and numerical experiments by the finite difference method are presented in Fujita-Katsurada-Kobari-Nagasaka [8], Fujita-Saito [10] and Fujita-Fukuhara-Saito [9]. These papers dealt only with the Poisson equation. This paper is sequel of these works and devoted to the similar problem for the Stokes equations. Namely, the purpose of the present paper is to study the domain-dependence of the convergence speed of some DDM for the Stokes equations.

We emphasize that this paper deal with the problem in continuous variables and does not consider the particular discrete problem. We believe that this enable us to find the analytical essence of our problem which does not depend on the particular discretization manner.

With a view to clarifying the nature of our results and the argument of our method, we restrict our consideration to a simple situation. Namely, we divide the whole domain into only two disjoint subdomains and deal only with a certain typical iterative scheme. Actually, we can prove that the convergence speed of such iteration depends upon the following:

(i) the relative relationship between subdomains described in terms of the reflection of subdomains with respect to an artificial boundary. These relationship is introduced as Condition (I);

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(ii) the characteristic quantity on the shape of the domain, say, the inf-sup constant of each subdomain.

In other words, we derive explicit decay rates of the error on an artificial boundary with the aid of Condition (I) and the inf-sup constants of two subdomains. Consequently, moreover, we can also get a concrete information on the choice of the relaxation parameter.

Our method of analysis is an extension of Fujita's one. Thus, it is based on (1) the theory of self-adjoint operators and their fractional powers applied to the Steklov-Poicaré operator for the Stokes equations and (2) the variational principle for the Stokes and the Laplace equations. However, it should be also noted that the description of our theorem does not involve any operator-theoretical concepts at all. Furthermore, apart from a treatment of our problem by operator theory, an investigation of the Stokes flow in *symmetric* domains plays a key role. There a new important role of the so-called inf-sup constant is revealed.

The results of this paper were already announced with a sketch of the proof in Saito [23]; the complete proof is provided in the present paper.

This paper is composed of seven sections. In Section 1, we formulate our problem. Specifically, we introduce the target problem, the way of dividing of the whole domain and the iterative scheme to be considered (the Dirichlet-Neumann (DN) iterations). Our main results are stated in Section 2, after having introduced some symbols and our shape condition, Condition (I). Section 3 presents some preliminary considerations concerning (1) function spaces, (2) trace theorems in domains with corners, and (3) solutions of the homogeneous Stokes equations. In Section 4, the proof of our theorem is established. We shall conclude the present paper with several additional remarks including a numerical example given in Section 5.

Remarks on the notaion. (1) Concerning function spaces and their norms, we follow the notation of Lions-Magenes [17]. We use the same symbols to denote a scalar-valued function space and a vector-valued one, whenever there is no possibility of confusion.
(2) By γ_0 we denote the trace operator from $H^1(\Omega)$ to $H^{1/2}(\gamma)$ and the boundary value $\gamma_0 v$ of $v \in H^1(\Omega)$ will be written conveniently as $v|_\gamma$. The meaning of $v|_{\Gamma_i}$, $i = 1, 2$, is similar.
(3) The precise definition of symbols in this paper will appear in Section 3.

1. FORMULATION OF THE PROBLEM

1.1. Target problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a piecewise smooth boundary Γ. Then our *target problem* is the following boundary value problem for the Stokes equations:

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = b & \text{on } \Gamma . \end{cases} \tag{1.1}$$

As usual, the vector function u denotes the flow velocity, the scalar function p the pressure. The prescribed vector function f defined in Ω and b on Γ are the external forces acting on the fluid and the boundary value, respectively. Throughout this paper, we assume that $f \in L^2(\Omega)$ and $b \in H^{1/2}(\Gamma)$. We assume in addition that the flux condition:

$$\int_\Gamma b \cdot n \, d\Gamma = 0 \tag{1.2}$$

is satisfied. Here n stands for the unit outer normal to Γ. By $\tilde{u}$ we denote the exact solution of (1.1) and by $\tilde{p}$ an accompanying pressure of the velocity $\tilde{u}$. Under assumptions above, unique existence of $\tilde{u} \in H^1(\Omega)$ is guaranteed on account of the standard theory of variational method. Besides it is well-known that $\tilde{p} \in L^2(\Omega)$ is uniquely determined up to an additive constant.

1.2. Decomposition of target domain

We decompose Ω into two disjoint subdomains Ω_1 and Ω_2 by a smooth simple curve γ;

$$\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2 \cup \gamma}, \qquad \Omega_1 \cap \Omega_2 = \emptyset .$$

We assume that γ connects transversally two points on Γ. Put $\Gamma_1 = \partial\Omega_1 \backslash \gamma$ and $\Gamma_2 = \partial\Omega_2 \backslash \gamma$. The unit outward normal to the boundary of a domain in consideration is denoted by n. If necessary, by ν we denote the one to γ outgoing from Ω_1. The curve γ is called the *artificial boundary*. See, for example, Fig. 1.

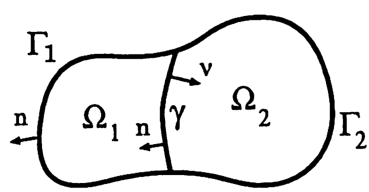

Figure 1. Decomposition of the target domain.

1.3. Iteration to be considered

Together with the above decomposition, we consider the following iterative scheme.

The Dirichlet-Neumann (DN) iterations. Let θ be the relaxation parameter such that $0 < \theta \le 1$. Take a vector-valued function $\mu^{(0)}$ on γ satisfying

$$\int_{\gamma} \mu^{(0)} \cdot \nu \, d\gamma + \int_{\Gamma_1} b \cdot n \, d\Gamma = 0 . \tag{1.3}$$

Then we successively generate $\{u_1^{(k)}, p_1^{(k)}\}$, $\{u_2^{(k)}, p_2^{(k)}\}$ and $\{\mu^{(k+1)}\}$, $k = 0, 1, 2, \ldots$, through

$$\begin{cases} -\Delta u_1^{(k)} + \nabla p_1^{(k)} = f & \text{in } \Omega_1 \\ \operatorname{div} u_1^{(k)} = 0 & \text{in } \Omega_1 \\ u_1^{(k)} = b & \text{on } \Gamma_1 \\ u_1^{(k)} = \mu^{(k)} & \text{on } \gamma \end{cases} \tag{1.4}$$

$$\begin{cases} -\Delta u_2^{(k)} + \nabla p_2^{(k)} = f & \text{in } \Omega_2 \\ \operatorname{div} u_2^{(k)} = 0 & \text{in } \Omega_2 \\ u_2^{(k)} = b & \text{on } \Gamma_2 \\ \dfrac{\partial u_2^{(k)}}{\partial n} - p_2^{(k)} n = -\dfrac{\partial u_1^{(k)}}{\partial \nu} + p_1^{(k)} \nu & \text{on } \gamma \end{cases} \tag{1.5}$$

$$\mu^{(k+1)} = (1-\theta)\mu^{(k)} + \theta u_2^{(k)}|_{\gamma} . \tag{1.6}$$

Several remarks are in order.

Remark 1.1. In the above (1.5), we have used the notation as

$$\frac{\partial u_1^{(k)}}{\partial \nu} - p_1^{(k)}\nu = [\nabla u_1^{(k)}]\nu - p_1^{(k)}\nu \tag{1.7}$$

where $[\nabla u_1^{(k)}]$ stands for the tensor-valued function of a vector-valued function $u_1^{(k)}$. Specifically, the first term of the right-hand side of (1.7) means the product of a tensor and a vector, and the second term the product of a scalar and a vector. As concerns the boundary condition of this type, the further remark will be mentioned in Section 5.

Remark 1.2. In view of (1.2) and the continuity condition in Ω_2, it is easy to verify that

$$\int_\gamma \mu^{(k)} \cdot \nu \,\mathrm{d}\gamma + \int_{\Gamma_1} b \cdot n \,\mathrm{d}\Gamma = 0, \qquad k = 0, 1, 2, \dots .$$

That is, the flux condition in Ω_1 is satisfied for each iteration.

Remark 1.3. Let an arbitrary $k \geq 0$ be fixed. An accompanying pressure $p_1^{(k)}$ of the velocity $u_1^{(k)}$ is uniquely determined up to an additive constant $c^{(k)}$. Hence, we can determine $c^{(k)}$ by standard techniques, for example, take $c^{(k)}$ such that $\int_{\Omega_1} p_1^{(k)}\,dx = 0$, or such that $p_1^{(k)} = 0$ at some point in Ω_1. Then the accompanying pressure $p_2^{(k)}$ of $u_2^{(k)}$ is uniquely determined. This means that $p_2^{(k)}$ depends on the choice of $p_1^{(k)}$.

Remark 1.4. Marini-Quarteroni [18] considered the discrete version of the DN iterations above by the finite element method and succeeded in deriving the following results: there exist two constants $0 < \theta_0 < \theta_1 < 1$ such that, if $\theta_0 < \theta < \theta_1$, it holds that

$$\|\nabla(\tilde{u}_h - u_{1,h}^{(k)})\|_{L^2(\Omega_1)} \leq R(\theta)^k \|\nabla(\tilde{u}_h - u_{1,h}^{(0)})\|_{L^2(\Omega_1)}, \quad k = 1, 2, 3, \dots$$

for some suitable constants $0 < R(\theta) < 1$. Moreover θ_0, θ_1 and $R(\theta)$ are independent of the mesh parameter h. Here, for instance, $\tilde{u}_h$ is a finite element counterpart of $\tilde{u}$. Their results only assert the existence of θ_0, θ_1 and $R(\theta)$. However, as mentioned in Introduction, we are interested in concrete relationships between convergence rates of the DN iterations and the way of decomposition. In other words, for the continuous problem, we are going to derive explicit values of convergence factor (convergence speed) and admissible bounds of the relaxation parameter θ.

Remark 1.5. As is well-known, under some suitable regularity assumptions, the problem (1.1) is equivalent to the muluti-domain problem:

$$\begin{cases} -\Delta u_1 + \nabla p_1 = f & \text{in } \Omega_1 \\ \operatorname{div} u_1 = 0 & \text{in } \Omega_1 \\ u_1 = b & \text{on } \Gamma_1, \end{cases} \qquad \begin{cases} -\Delta u_2 + \nabla p_2 = f & \text{in } \Omega_2 \\ \operatorname{div} u_2 = 0 & \text{in } \Omega_2 \\ u_2 = b & \text{on } \Gamma_2 \end{cases}$$

with the interface conditions

$$\begin{cases} u_1 = u_2 & \text{on } \gamma \\ \dfrac{\partial u_1}{\partial \nu} - p_1\nu = -\dfrac{\partial u_2}{\partial n} + p_2 n & \text{on } \gamma. \end{cases}$$

Moreover this muluti-domain problem is reduced to the interface equation:

$$\Sigma\mu = \chi, \qquad \mu = \tilde{u}|_\gamma \tag{1.8}$$

where Σ denotes an operator acting on functions on γ, and χ a function defined on γ. Actually, the DN iterations is regarded as a Richardson-type iteration on (1.8) using a certain preconditioner. On the other hand, there are several strategies to precondition Σ, and they correspond with domain decomposition iterative schemes. Our description here is somewhat formal; We can obtain more detailed and precise discussions about such aspects of the domain decomposition methods in the monograph [22] by A. Quarteroni and A. Valli.

2. MAIN RESULTS

In this section, our main results are stated. To this end, we need some symbols. We confine our attention to the error $\xi^{(k)}$ on γ which is defined by

$$\xi^{(k)} = \mu^{(k)} - \tilde{u}|_\gamma .$$

Denote V a vector-valued function space $V = \{\xi \in H^{1/2}(\gamma);\, \|\xi\|_V < \infty\}$ with

$$\|\xi\|_V = \left\{ \|\xi\|^2_{H^{1/2}(\gamma)} + \left\|\rho^{-1/2}\xi\right\|^2_{L^2(\gamma)} \right\}^{1/2}$$

where ρ is the distance from the end points of γ. On the other hand, it is well-known that there exists a positive constant satisfying

$$\inf_{q\in L^2_0(\Omega_i)} \sup_{v\in H^1_0(\Omega_i)} \frac{(q, \operatorname{div} v)_i}{\|q\|_i \, \|\nabla v\|_i} = \beta_i, \qquad i = 1,2 .$$

Here we have employed the following notation:

- $(u, v)_i =$ a usual $L^2(\Omega_i)$ inner product, and $\|u\|_i = (u,u)_i^{1/2}$;
- $\|\nabla v\|_i^2 = \sum_{1\le m,n\le 2} \|\partial v^m/\partial x_n\|_i^2$ for $v = (v^1, v^2) \in H^1(\Omega_i)$;
- $L^2_0(\Omega_i) = \{q \in L^2(\Omega_i);\ (q,1)_i = 0\}$.

The constant β_i is called the inf-sup constant corresponding to Ω_i and introduced independently by Babuška [2] and Brezzi [5]. When Ω_i is a square, we have

$$\beta_i^{-1} \le (4 + 2\sqrt{2})^{1/2} = 2.6131\ldots . \tag{2.1}$$

This can be derived by combining the result in Horgan-Payne [15] with that in Velte [26]. The additional information on the inf-sup constant will be given in Section 5.

At this stage, we introduce our shape condition which is firstly considered by Fujita [6].

Condition (I). Let γ be a line segment on the x_2-axis and let Ω_2' be the image of Ω_2 by reflection with respect to the x_2-axis. Then, (Ω, γ) is said to satisfy Condition (I) if an inclusion $\Omega_2' \subseteq \Omega_1$ holds. See, for instance, Fig. 2.

Now we can state our main theorem.

Theorem 2.1. *Put* $\alpha = 1 + (1 + \beta^{-1})^2$, $\beta = \max(\beta_1, \beta_2)$ *and, for* $0 < \theta < 2/\alpha$, *define*

$$\tilde{r}(\theta) = \begin{cases} 1 - \theta & \text{for } 0 < \theta \le 2/(\alpha+1) \\ \alpha\theta - 1 & \text{for } 2/(\alpha+1) \le \theta < 2/\alpha . \end{cases} \tag{2.2}$$

Suppose that (Ω, γ) *satisfies Condition* (I). *Then there exists a positive constant* c_0 *depending only on* (Ω, γ) *such that*

$$\|\xi^{(k)}\|_V \le c_0 \tilde{r}^k \|\xi^{(0)}\|_V, \qquad k = 1, 2, 3, \ldots \tag{2.3}$$

holds true.

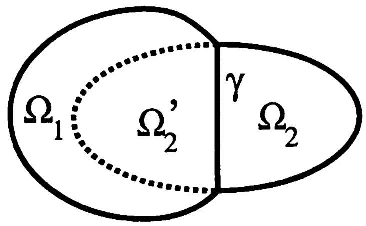

Figure 2. Example of geometry satisfying Condition (I).

Remark 2.1. Under the same assumptions of Theorem 2.1, taking $\theta = \theta_o \equiv 2/(\alpha + 1)$, we get the minimum $\tilde{r}_{\min} = (\alpha - 1)/(\alpha + 1)$ of $\tilde{r}(\theta)$. In other words, the value which brings about the fastest decay speed is smaller than or equals to $\tilde{r}_{\min}$. That is, we could say that $\theta = \theta_o$ is the optimal choice of θ among all values of θ which yield the exponential decay of the error and is universally optimal for *any* (Ω, γ) *as long as Condition* (I) *is satisfied.*

These theorems, together with (2.1), imply the following corollary.

Corollary 2.1. *Suppose that* (Ω, γ) *satisfies Condition* (I) *and moreover that both* Ω_1 *and* Ω_2 *are squares. Then, as long as* $0 < \theta < 0.1423$, *the exponential decay of the error as stated in* (2.3) *is guaranteed. Furthermore, in this case, we get* $\tilde{r}_{\min} \le 0.8671$.

3. PRELIMINARIES

This section presents preliminary considerations which we need to prove our theorem. In Subsection 3.1, we collect function spaces. Subsection 3.2 is devoted to auxiliary lemmas concerning the trace theorem. In Subsection 3.3, we are concerned with solutions of the homogeneous Stokes equations. In order to discuss in a general context, for the time being, we assume that

$$\begin{cases} \Omega \text{ is an arbitrary bounded domain in } \mathbb{R}^2; \\ \text{a smooth curve } \gamma \text{ is a part of } \partial\Omega; \\ \Gamma = \partial\Omega \backslash \gamma \text{ is a piecewise smooth curve;} \\ n \text{ is outward normal on } \gamma; \\ \gamma \text{ connects } \Gamma \text{ at two points;} \\ \text{intersections of } \gamma \text{ and } \overline{\Gamma} \text{ are corner points of } \Omega \\ \quad \text{which are not turning points.} \end{cases} \tag{3.1}$$

3.1. Function spaces

Below we collect function spaces we will use. Although a few of them are already appeared, for the convenience of later reference, we again state their definitions.

(i) $X =$ a vector-valued $L^2(\gamma)$. $(\xi, \eta)_X$ and $\|\xi\|_X$ denote $L^2(\gamma)$ inner product and norm, respectively;

(ii) $X_\sigma = \{\xi \in X;\ (\xi, n)_X = 0\}$ which is *our basic Hilbert space*;

(iii) $V_0 = \{\xi \in C_0^\infty(\gamma);\ (\xi, n)_X = 0\}$;

(iv) $V = \{\xi \in H^{1/2}(\gamma);\ \|\xi\|_V < \infty\}$ with the norm

$$\|\xi\|_V = \left\{\|\xi\|^2_{H^{1/2}(\gamma)} + \|\rho^{-1/2}\xi\|^2_{L^2(\gamma)}\right\}^{1/2}$$

where ρ stands for the distance from the end points of γ. It is well-known that V is a Hilbert space equipped with a norm $\|\xi\|_V$;

(v) $V_\sigma = V \cap X_\sigma$ which is a closed subspace of V;

(vi) $L_0^2(\Omega) = \{q \in L^2(\Omega);\ (q, 1)_{L^2(\Omega)} = 0\}$;

(vii) $H_0^1(\Omega) = \{v \in H^1(\Omega);\ v|_{\partial\Omega} = 0\}$;

(viii) $H_{0,\sigma}^1(\Omega) = \{v \in H_0^1(\Omega);\ \operatorname{div} v = 0 \text{ in } \Omega\}$;

(ix) $K^1(\Omega) = \{v \in H^1(\Omega);\ v|_\Gamma = 0\}$;

(x) $K_\sigma^1(\Omega) = \{v \in K^1(\Omega);\ \operatorname{div} v = 0 \text{ in } \Omega\}$.

3.2. Trace theorem and Stokes extension

We introduce a continuous bilinear form on $K^1(\Omega)$

$$a(u,v) = a_\Omega(u,v) = \int_\Omega \nabla u \nabla v \, dx = \sum_{1\le i,j\le 2} \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, dx\,. \tag{3.2}$$

It is evident that a is $K^1(\Omega)$-elliptic, i.e., there exists a positive constant C depending only on Ω such that

$$a(u,u) \ge C\|u\|^2_{H^1(\Omega)} \quad \forall u \in K^1(\Omega)\,. \tag{3.3}$$

Remark 3.1. In what follows, the symbol C denotes a positive constant which depends only on Ω. The value of C may change even in the same context. The meaning of the symbol C' is same.

Below, we will deal with the following boundary value problem:

$$\begin{cases} \Delta w - \nabla p = 0 & \text{in } \Omega \\ \operatorname{div} w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \Gamma \\ w = \xi & \text{on } \gamma\,. \end{cases} \tag{3.4}$$

We make the meanings of the problem (3.4) and its solution more precise. We begin by the following lemma concerning the trace on γ of a function in $H^1(\Omega)$.

Lemma 3.1. *The following two assertions hold true.*

(i) *For every $\psi \in K^1(\Omega)$, we have $\eta = \psi|_\gamma \in V$ and $\|\eta\|_V \le C\|\psi\|_{H^1(\Omega)}$.*

(ii) *For every $\eta \in V$, there exists a $\psi \in K^1(\Omega)$ such that $\psi|_\gamma = \eta$ and $\|\psi\|_{H^1(\Omega)} \le C'\|\eta\|_V$.*

Remark 3.2. For the proof of Lemma 3.1, we refer to Grisvard [13]. Assume that γ is a line segment, and suppose that L is a minus Laplacian on γ with a zero Dirichlet boundary condition. Then, as a matter of fact, V is coincident algebraically and topologically with $\mathfrak{D}(L^{1/4})$ which denotes the domain of a 1/4-power of L (Fujiwara [11]). According to this, we can give an elementary proof of Lemma 3.1 by using the eigenvalues and the eigenfunctions of L (see Saito-Fujita [25]).

The solenoidal (div $v = 0$) version of the above lemma holds also. That is, we have the following statement.

Lemma 3.2. *The following two assertions hold true.*

(i) *For every $v \in K^1_\sigma(\Omega)$, we have $\xi = v|_\gamma \in V_\sigma$ and $\|\xi\|_V \le C\|v\|_{H^1(\Omega)}$.*

(ii) *For every $\xi \in V_\sigma$, there exists a $v \in K^1_\sigma(\Omega)$ such that $v|_\gamma = \xi$ and $\|v\|_{H^1(\Omega)} \le C'\|\xi\|_V$.*

Proof. (i) This is immediately obtained by virtue of Lemma 3.1 (i) and the Gauss divergence theorem

$$0 = \int_\Omega \operatorname{div} v \, dx = \int_{\partial\Omega} v \cdot n \, d\Gamma = \int_\gamma \xi \cdot n \, d\gamma .$$

(ii) We employ a standard argument (see, for example, in Arnold-Scott-Vogelius [1]). Let $\xi \in V_\sigma$, and suppose that $\psi \in K^1(\Omega)$ is an extension of ξ as in Lemma 3.1 (ii). We introduce a scalar function $F \in L^2(\Omega)$ by setting $F = -\operatorname{div} \psi$. Then we have $F \in L^2_0(\Omega)$ since

$$\int_\Omega F \, dx = -\int_\Omega \operatorname{div} \psi \, dx = -\int_\gamma \xi \cdot n \, d\gamma = 0 .$$

Therefore, by virtue of the Babuška-Aziz lemma (Lemma 5.4.2, [3]), there exists a vector function $\zeta \in H^1_0(\Omega)$ such that $\operatorname{div} \zeta = F$ in Ω and $\|\nabla\zeta\|_{L^2(\Omega)} \le C\|F\|_{L^2(\Omega)}$. Then, we put $v = \psi + \zeta$ which is the desired function. □

Remark 3.3. When Ω is a polygon, a generalization of Lemma 3.2 is presented in Arnold-Scott-Vogelius [1].

At this stage, we can describe the definition of the Stokes extension which is already introduced in the finite dimensional case in Marini-Quarteroni [18] and Quarteroni [19]. According to Lemma 3.2, for any $\xi \in V_\sigma$, the set

$$K^1_{\sigma,\xi}(\Omega) = \{v \in K^1_\sigma(\Omega);\ v|_\gamma = \xi\} \tag{3.5}$$

is not empty. Therefore, by virtue of the standard theory of variational methods, for any $\xi \in V_\sigma$, the problem (3.4) admits of a solution $\{w, p\}$. More precise, there exist $w \in K^1_\sigma(\Omega)$ and $p \in L^2(\Omega)$ satisfying

$$a(w, \varphi) - (p, \operatorname{div} \varphi)_{L^2(\Omega)} = 0 \quad \forall \varphi \in H^1_0(\Omega), \qquad w|_\gamma = \xi . \tag{3.6}$$

As is well-known, w is unique and p is uniquely determined up to an additive constant. We call w the *Stokes extension* of ξ into Ω and p an accompanying pressure of w.

The following lemma is an easy consequence of Lemma 3.2 and (3.6).

Lemma 3.3. *The inequality*

$$C\|\xi\|_V^2 \le a(w, w) \le C'\|\xi\|_V^2 \tag{3.7}$$

holds for any $\xi \in V_\sigma$. Here w is the Stokes extension of ξ into Ω. Namely, the norm $a(w, w)^{1/2}$ is equivalent to $\|\xi\|_V$ in V_σ.

Remark 3.4. An extensive analysis concerning the Stokes extension and the harmonic extension described below are presented in Quarteroni-Valli [22].

3.3. Symmetry of domains and the Stokes flows

This subsection is devoted to a study on properties of the Stokes extensions. Let $\xi \in V$. Then the solution $g \in K^1(\Omega)$ of the harmonic problem:

$$\Delta g = 0 \text{ in } \Omega, \quad g = 0 \text{ on } \Gamma, \quad g = \xi \text{ on } \gamma \tag{3.8}$$

is the *harmonic extension* of ξ into Ω.

Lemma 3.4. *Let $\xi \in V_\sigma$, and suppose that $w \in K^1_\sigma(\Omega)$ and $g \in K^1(\Omega)$ are the Stokes extension and the harmonic extension of ξ into Ω, respectively. Moreover let β be the inf-sup constant corresponding to Ω. Then we have*

$$a(g,g) \le a(w,w) \le (1+\beta^{-1})^2 a(g,g)\,. \tag{3.9}$$

Proof. The first inequality is deduced from the variational principle for the harmonic function;

$$a(g,g) = \min\{a(v,v);\ v \in K^1(\Omega), v = \xi \text{ on } \gamma\}\,. \tag{3.10}$$

The second inequality is essentially obtained in Marini-Quarteroni [18]. For the completeness of the argument, however, we state its proof. For the sake of simplicity, we write $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$ and $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. Let $p \in L^2_0(\Omega)$ be the accompanying pressure of w. Then it holds from the definition of β that

$$\beta\|p\| \le \sup_{v \in H^1_0(\Omega)} \frac{(p, \operatorname{div} v)}{\|\nabla v\|} = \sup_{v \in H^1_0(\Omega)} \frac{(\nabla w, \nabla v)}{\|\nabla v\|} \le \|\nabla w\|\,.$$

As a result, we deduce

$$\begin{aligned} a(w,w) &= a(w, w-g) + a(w,g) = (p, \operatorname{div}(w-g)) + a(w,g) \\ &\le \|p\|\,\|\nabla g\| + \|\nabla w\|\,\|\nabla g\| \le \Big(1+\frac{1}{\beta}\Big)\|\nabla w\|\|\nabla g\|\,. \end{aligned}$$

Hence we arrive at

$$\|\nabla w\| \le \Big(1+\frac{1}{\beta}\Big)\|\nabla g\|\,.$$

This completes the proof. □

Now we can prove the following lemma which plays a key role in the proof of our theorem.

Lemma 3.5. *Assume that both Ω_1 and Ω_2 are two bounded domains and share a part γ of their boundaries. Moreover, assume that γ is a line segment on the x_2-axis and that Ω_1 and Ω_2 are reflection of each other with respect to γ. Put $\beta = \beta_1 = \beta_2$, $a_1(\cdot,\cdot) = a_{\Omega_1}(\cdot,\cdot)$ and $a_2(\cdot,\cdot) = a_{\Omega_2}(\cdot,\cdot)$. Then it holds that*

$$(1+\beta^{-1})^{-2} a_2(w_2,w_2) \le a_1(w_1,w_1) \le (1+\beta^{-1})^2 a_2(w_2,w_2) \tag{3.11}$$

for any $\xi \in V_\sigma$, where $w_1 \in K^1_\sigma(\Omega_1)$ and $w_2 \in K^1_\sigma(\Omega_2)$ are the Stokes extensions of ξ into Ω_1 and Ω_2, respectively.

Proof. Let $g_1 \in K^1(\Omega_1)$ and $g_2 \in K^1(\Omega_2)$ be the harmonic extensions of ξ into Ω_1 and Ω_2, respectively. It is clear that $g_1(x_1,x_2) = g_2(-x_1,x_2)$ and therefore $a_1(g_1,g_1) = a_2(g_2,g_2)$. This equality, together with Lemma 3.4, implies (3.11). □

4. PROOF OF THEOREM

This section establishes the proof of Theorem 2.1. To this end, we adopt Fujita's method [6, 7] with necessary modifications to fit the present problem. In fact, the proof will be accomplished along the following scheme: (1) derivation of a recursive expression of the error in terms of the amplification operator (Subsection 4.3); (2) introduction of the special inner product in the function space under consideration to treat the amplification operator as a self-adjoint operator (Subsection 4.2); (3) estimation of the spectrum radius of the amplification operator under our shape condition (Subsection 4.3 and 4.4). At the stages (1), (2), we use the so-called Steklov-Poincaré operator for the Stokes equations, which is defined in Subsection 4.1. That is, the amplification operator and the special inner product are defined in terms of the square root of the Steklov-Poincaré operator.

Notation concerning operator theory. For an operator T, we write $\mathfrak{D}(T)$ and $\mathfrak{R}(T)$ to indicate the domain and the range of T, respectively. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two Hilbert spaces. By $\mathfrak{L}(\mathfrak{X}, \mathfrak{Y})$ we denote the space of all linear continuous operators from $\mathfrak{X}$ into $\mathfrak{Y}$, with the operator norm

$$\|T\| = \|T\|_{\mathfrak{L}(X_1,X_2)} = \sup_{u\in\mathfrak{X}} \frac{\|Tu\|_{\mathfrak{Y}}}{\|u\|_{\mathfrak{X}}}.$$

In addition, we write as $\mathfrak{L}(\mathfrak{X}) = \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$.

4.1. Steklov-Poincaré operator

In this subsection, we are going to define the Steklov-Poincaré operator for the Stokes equations through the theory of unbounded quadratic form in a Hilbert space by Kato [16], which we call Kato's theory.

We assume that (3.1) holds. For the time being, we present formal discussions. We assume that, for $\xi \in V_0$, a *smooth vector function* w and a *smooth scalar function* p solve the homogeneous Stokes equations (3.4). Then an operator S_0 is defined by

$$S_0\xi = \left(\frac{\partial w}{\partial n} - pn\right)\Big|_\gamma .$$

We would like to deal with S_0 as an operator of $X_\sigma \to X_\sigma$. This is possible by controlling an additive constant of an accompanying pressure p of w. In fact, let $p_0 \in L^2_0(\Omega)$ be the accompanying pressure of w and put $p = p_0 + \kappa$ with a constant κ. Then we have

$$(S_0\xi, n)_X = \int_\gamma \left(\frac{\partial w}{\partial n} - p_0 n\right) \cdot n \, \mathrm{d}\gamma - \kappa|\gamma|$$

where $|\gamma|$ is the measure of γ. Therefore, if we take

$$\kappa = \frac{1}{|\gamma|} \int_\gamma \left(\frac{\partial w}{\partial n} - p_0 n\right) \cdot n \, \mathrm{d}\gamma \tag{4.1}$$

we ensure that $S_0\xi \in X_\sigma$.

Actually, the operator S_0 is a primitive version of the Steklov-Poincaré operator that we want to define. In general, unfortunately, even if ξ is taken from V_0, its Stokes extension w and the accompanying pressure p do not have regularities that justify our consideration above. This is the reason that we employ Kato's theory, which needs no regularity assumptions on w but $w \in H^1(\Omega)$.

Now we proceed to a rigorous definition. Let us introduce a quasilinear form J in X_σ by

$$\begin{cases} \text{the domain } \mathfrak{D}(J) \text{ of } J \text{ is } V_\sigma \\ J[\xi,\eta] = a(w,v) \qquad \forall\, \xi,\eta \in \mathfrak{D}(J) \end{cases}$$

where w and v are the Stokes extensions of ξ and η into Ω, respectively. We shall not distinguish J from the corresponding quadratic form $J[\xi] = J[\xi,\xi]$. Obviously, J is positive and symmetric. Moreover, we deduce the following lemma.

Lemma 4.1. *J is a closed form in Kato's sense. Namely,*

$$\xi_n \in \mathfrak{D}(J), \quad \xi_n \to \xi_0 \text{ in } X_\sigma, \quad J[\xi_n - \xi_m] \to 0\,, \quad n,m \to \infty \tag{4.2}$$

implies that

$$\xi_0 \in \mathfrak{D}(J), \quad J[\xi_n - \xi_0] \to 0\,, \quad n \to \infty\,. \tag{4.3}$$

Proof. Assume that (4.2) holds . Then it holds from (3.7) that $\{\xi_n\}$ is a Cauchy sequence in V_σ. Since V_σ is complete, there exists a $\xi_0^* \in V_\sigma = \mathfrak{D}(J)$ such that

$$J[\xi_n - \xi_0^*] \to 0\,, \quad n \to \infty\,.$$

By the uniqueness of the limit, $\xi_0^* = \xi_0$ so that (4.3) follows. □

Lemma 4.2 (the Steklov-Poincaré operator). *There exists an operator $S : \mathfrak{D}(S) \subset X_\sigma \to X_\sigma$ associated with J such that*

(i) *S is positive and self-adjoint in X_σ. $S^{1/2}$ is also self-adjoint in X_σ;*

(ii) *$\mathfrak{D}(S) \subset \mathfrak{D}(J) = V_\sigma$ and $J[\xi,\eta] = (S\xi,\eta)_X \ \forall\, \xi \in \mathfrak{D}(S), \forall\, \eta \in V_\sigma$;*

(iii) *$\mathfrak{D}(S^{1/2}) = V_\sigma$ and $J[\xi,\eta] = (S^{1/2}\xi, S^{1/2}\eta)_X \ \forall\, \xi,\eta \in V_\sigma$;*

(iv) *$S^{1/2} \in \mathfrak{L}(V_\sigma, X_\sigma)$ and $S^{-1/2} \in \mathfrak{L}(X_\sigma, V_\sigma)$.*

The operator S is uniquely determined by the condition (ii).

Proof. Applying J to the representation theorems (Chapter VI-§2, Kato [16]), we immediately obtain (i)-(iii). Since it holds from (3.7) and (iii) that

$$C\|\xi\|_V \le \|S^{1/2}\xi\|_X \le C'\|\xi\|_V, \qquad \xi \in V_\sigma \tag{4.4}$$

we have $S^{1/2} \in \mathfrak{L}(V_\sigma, X_\sigma)$. This implies that $\mathfrak{R}(S^{1/2})$ is closed, and therefore, by virtue of the density of $\mathfrak{R}(S^{1/2})$ in X_σ, we have $\mathfrak{R}(S^{1/2}) = X_\sigma$. It is also derived from (4.4) that $S^{1/2}$ is bijective. Hence we deduce (iv). □

The operator S will be called the *Steklov-Poincaré operator for the Stokes equations* (or simply the *SPS operator*) pertaining to (Ω,γ). We shall often write $S = S(\Omega,\gamma)$ to express this correspondence.

Remark 4.1. If ξ and its Stokes extension w have suitable regularities, in view of Lemma 4.2, we obtain

$$(S\xi,\xi)_X = a(w,w) = \int_\gamma \Big(\frac{\partial w}{\partial n} - pn\Big)\xi \,\mathrm{d}\gamma$$

where the accompanying pressure p has been chosen subject to (4.1). This equality means that

$$S\xi = \left(\frac{\partial w}{\partial n} - pn\right)\Big|_\gamma$$

holds in the distribution sense.

Remark 4.2. The concrete characterization of $\mathfrak{D}(S)$ is an open problem.

The operator $S^{1/2}$ can be characterized by the variational principle. In fact, for any $\xi \in V_\sigma$, the variational principle for the Stokes equation claims that

$$a(w,w) \leq a(v,v) \qquad \forall\, v \in K^1_{\sigma,\xi}(\Omega)$$

where w is the Stokes extension of ξ. This, together with Lemma 4.2 (iii), implies

$$\|S^{1/2}\xi\|_X^2 \leq a(v,v) \qquad \forall\, v \in K^1_{\sigma,\xi}(\Omega)\,.$$

4.2. New inner product in V_σ

We consider the case where

$$\begin{cases} \Omega_1, \Omega_2 \text{ are arbitrary bounded domains and} \\ \text{share the interface boundary } \gamma \text{ and } \Gamma_1 = \partial\Omega_1\backslash\gamma,\ \Gamma_2 = \partial\Omega_2\backslash\gamma\,. \end{cases}$$

And write

$$S_1 = S(\Omega_1, \gamma) \quad \text{and} \quad S_2 = S(\Omega_2, \gamma)\,.$$

We here investigate the operator $S_2^{-1}S_1$ which plays an important role in the argument below.

Lemma 4.3. *The operator $H_0 = S_2^{-1}S_1$ admits of a bounded extension in V_σ. Moreover its extension is given by*

$$H = S_2^{-1/2}(S_1^{1/2}S_2^{-1/2})^* S_1^{1/2} \tag{4.5}$$

where $(\cdot)^*$ means the adjoint operator in X_σ.

Proof. It is easy to verify that $H_0\xi = H\xi$ for $\xi \in \mathfrak{D}(H_0) = \mathfrak{D}(S_1)$. On the other hand, the boundness of H follows from

$$S_2^{-1/2} \in \mathfrak{L}(X_\sigma, V_\sigma), \quad (S_1^{1/2}S_2^{-1/2})^* \in \mathfrak{L}(X_\sigma), \quad S_1^{1/2} \in \mathfrak{L}(V_\sigma, X_\sigma).$$

In fact, since it is clear that $S_1^{1/2}S_2^{-1/2} \in \mathfrak{L}(X_\sigma)$, its adjoint operator $(S_1^{1/2}S_2^{-1/2})^*$ is also bounded in X_σ. □

The crucial point of our method is to introduce a special inner product in V_σ in terms of the square root of the SPS operators as follows:

$$((\xi, \eta)) = (S_2^{1/2}\xi, S_2^{1/2}\eta)_X \qquad \forall\, \xi, \eta \in V_\sigma\,. \tag{4.6}$$

Then it follows from the linearity of $S_2^{1/2}$ that V_σ again forms a Hilbert space with the new inner product (4.6). Moreover, in view of (3.7), we deduce that the corresponding norm $|||\xi||| = ((\xi,\xi))^{1/2}$ is equivalent to $\|\xi\|_V$ in V_σ.

Lemma 4.4. *Under the new inner product* (4.6), *the operator H defined by* (4.5) *is self-adjoint in V_σ.*

Proof. Since H is bounded in V_σ, it suffices to verify that H is symmetric. This is seen from

$$\begin{aligned}((H\xi,\eta)) &= ((S_1^{1/2}S_2^{-1/2})^*S_1^{1/2}\xi, S_2^{1/2}\eta)_X = (S_1^{1/2}\xi, S_1^{1/2}\eta)_X \\ ((\xi,H\eta)) &= (S_2^{1/2}\xi, (S_1^{1/2}S_2^{-1/2})^*S_1^{1/2}\eta)_X = (S_1^{1/2}\xi, S_1^{1/2}\eta)_X\end{aligned}$$

which completes the proof. □

H is not self-adjoint with the usual X_σ inner product. Thus, introducing the new inner product (4.6) allows us to treat H as a self-adjoint operator.

4.3. Amplification operator for the error

We return to the DN iterations. We keep the notation of Section 1. Let us introduce error functions as follows:

$$\begin{cases} v_1^{(k)} = u_1^{(k)} - \tilde{u}|_{\Omega_1}, & q_1^{(k)} = p_1^{(k)} - \tilde{p}|_{\Omega_1} \\ v_2^{(k)} = u_2^{(k)} - \tilde{u}|_{\Omega_2}, & q_2^{(k)} = p_2^{(k)} - \tilde{p}|_{\Omega_2}. \end{cases}$$

These functions solve

$$\begin{cases} \Delta v_1^{(k)} - \nabla q_1^{(k)} = 0 & \text{in } \Omega_1 \\ \operatorname{div} v_1^{(k)} = 0 & \text{in } \Omega_1 \\ v_1^{(k)} = 0 & \text{on } \Gamma_1 \\ v_1^{(k)} = \xi^{(k)} & \text{on } \gamma \end{cases}$$

$$\begin{cases} \Delta v_2^{(k)} - \nabla q_2^{(k)} = 0 & \text{in } \Omega_2 \\ \operatorname{div} v_2^{(k)} = 0 & \text{in } \Omega_2 \\ v_2^{(k)} = 0 & \text{on } \Gamma_2 \\ \dfrac{\partial v_2^{(k)}}{\partial n} - q_2^{(k)} n = -\dfrac{\partial v_1^{(k)}}{\partial \nu} + q_1^{(k)}\nu & \text{on } \gamma \end{cases}$$

$$\xi^{(k+1)} = (1-\theta)\xi^{(k)} + \theta v_2^{(k)}|_\gamma .$$

Therefore $\xi^{(k+1)}$ can be expressed, at least in the formal manner, as

$$\xi^{(k)} = (1-\theta)\xi^{(k)} - \theta S_2^{-1}S_1\xi^{(k)} .$$

In view of Lemmas 4.3 and 4.4, we can replace $S_2^{-1}S_1$ by H. Thus, by putting

$$A_\theta = (1-\theta)I - \theta H \tag{4.7}$$

we arrive at a recursive expression of $\xi^{(k)}$ of the form

$$\begin{cases} \xi^{(0)} \in V_\sigma \\ \xi^{(k+1)} = A_\theta \xi^{(k)}, \quad k = 0,1,2,\ldots . \end{cases} \tag{4.8}$$

This means that the error on γ is successively generated in accordance with (4.8). We will call A_θ the *amplification operator* for the error or the *error-generating operator.*

Since H is bounded, self-adjoint in V_σ with the special inner product (4.6), A_θ has the same properties too. By $r_\sigma(A_\theta)$ we denote the spectral radius of A_θ. Then, on account of (4.8), we have

$$|||\xi^{(k+1)}||| = r_\sigma(A_\theta)|||\xi^{(k)}|||, \quad k = 0, 1, 2, \ldots \tag{4.9}$$

where $|||\xi||| = ((\xi, \xi))^{1/2}$ and we have used the fact that A_θ is self-adjoint with (4.6).

Moreover, we have the following lemma.

Lemma 4.5. *Let $\sigma(H)$ be the spectrum of H. Then we have*

$$r_\sigma(A_\theta) = \sup_{\lambda \in \sigma(H)} |1 - \theta - \theta\lambda|. \tag{4.10}$$

Proof. This is a direct consequence of the spectral mapping theorem (see, for example, Yosida [27]). Namely, noting that $A_\theta = 1 - \theta - \theta H$, we deduce

$$r_\sigma(A_\theta) = \sup_{\zeta \in \sigma(A_\theta)} |\zeta| = \sup_{\lambda \in \sigma(H)} |1 - \theta - \theta\lambda|$$

where $\sigma(A_\theta)$ stands for the spectrum of A_θ. □

4.4. Spectrum of H and proof of theorems

By virtue of the argument in the previous subsection, if we obtain a concrete information on the spectrum of H, we can calculate the right-hand side of (4.10). Thus we have an estimation of the spectral radius of A_θ. Applying those estimation in conjunction with the equality (4.9), we obtain an estimation of $\xi^{(k)}$ in the $|||\cdot|||$-norm. We recall that Marini-Quarteroni [18] essentially proved that

$$C_1 \le \lambda \le C_2 \qquad \forall \lambda \in \sigma(H)$$

where C_i is a positive constant depending on Ω_i. Therefore, the estimation of $r_\sigma(A_\theta)$ in terms of C_1 and C_2 can be obtained. However, we want to know more explicit values of C_1 and C_2. This is possible with the aid of Condition (I). Thus, we can prove the following lemma.

Lemma 4.6. *Suppose that (Ω, γ) satisfies Condition* (I) *and put $\beta = \max(\beta_1, \beta_2)$. Here β_1 and β_2 are the inf-sup constants corresponding to Ω_1 and Ω_2, respectively. Then we have*

$$0 \le \lambda \le (1 + \beta^{-1})^2 \qquad \forall \lambda \in \sigma(H).$$

Proof. Let $\xi \in V_\sigma$ be fixed. Suppose that Ω_2' is the reflection of Ω_2 with respect to γ and $w_2' \in K_\sigma^1(\Omega_2')$ be the Stokes extension of ξ into Ω_2'. Put $a_1 = a_{\Omega_1}$, $a_2 = a_{\Omega_2}$ and $a_2' = a_{\Omega_2'}$. By the assumption, $\Omega_2' \subseteq \Omega_1$ holds. We introduce the zero extension v of w_2' into Ω_1;

$$v = \begin{cases} w_2' & \text{in } \Omega_2 \\ 0 & \text{in } \Omega_1 \backslash \Omega_2 . \end{cases}$$

By virtue of the variational principle for the Stokes equations, we have

$$a_1(w_1, w_1) \le a_1(v, v) = a_2'(w_2', w_2') .$$

Hence, from Lemma 3.5, we deduce

$$a_1(w_1, w_1) \le (1 + \beta_2^{-1})^2 a_2(w_2, w_2) . \tag{4.11}$$

On the other hand, let g_1, g_2 and g_2' be the harmonic extensions of ξ into Ω_1, Ω_2 and Ω_2', respectively. Then, in view of the variational principle for the harmonic function, it holds that

$$a_1(g_1, g_1) \le a_2'(g_2', g_2') = a_2(g_2, g_2).$$

Therefore, again from Lemma 3.5, we deduce

$$a_1(w_1, w_1) \le (1+\beta_1^{-1})^2 a_1(g_1, g_1) \le (1+\beta_1^{-1})^2 a_2(g_2, g_2) \le (1+\beta_1^{-1})^2 a_2(w_2, w_2). \quad (4.12)$$

This inequality, together with (4.11), implies

$$a_1(w_1, w_1) \le (1+\beta^{-1})^2 a_1(w_2, w_2). \quad (4.13)$$

Now note that

$$((H\xi, \xi)) = \|S_1^{1/2}\xi\|_X^2 = a_1(w_1, w_1), \quad |||\xi|||^2 = \|S_2^{1/2}\xi\|_X^2 = a_2(w_2, w_2).$$

Thus, (4.13) can be written as

$$((H\xi, \xi)) \le (1+\beta^{-1})^2 |||\xi|||^2$$

which completes the proof. □

We are now in a position to prove Theorem 2.1. Put $\alpha = 1 + (1+\beta^{-1})^2$. According to Lemma 4.5, we have

$$r_\sigma(A_\theta) \le \sup_{0\le\lambda\le\alpha-1} |1-\theta-\lambda\theta| = \max\{|1-\theta|, |1-\alpha\theta|\}.$$

The right-hand side of this inequality is nothing but $\tilde{r}(\theta)$ which is appeared in Theorem 2.1. This completes the proof.

5. CONCLUDING REMARKS

5.1. The discrete DN iterations

As mentioned in introduction, we restrict our consideration to the problem of continuous variables. However, from the practical point of view, we are interested in the corresponding discrete problem. Let us consider a discrete version of each subproblem in the DN iterations by the finite element method. Namely, we consider the discrete DN iterations which is studied in Marini-Quaeteroni [18]. Unfortunately, the theory developed in this paper can not be expected to be valid for the discrete DN iterations. In fact, Lemma 3.5 which plays a key role in our analysis is not true, if we consider the discrete problem, since the triangulations of subdomains are in general non-symmetric. Accordingly, if we want to obtain the similar results for the discrete DN iterations, we need to assume that the triangulations of Ω_1 and Ω_2 are symmetric with respect to the line segment γ. This is a quite strong restriction from the practical point of view. Therefore we have to make a new device to treat the discrete DN iterations, which will be discussed in another paper.

We present a numerical example to illustrate our theoretical results. To this end, we assume that:

$$\Omega_1 = \{-a < x_1 < 0,\ 0 < x_2 < a\}, \quad \Omega_2 = \{0 < x_1 < a,\ 0 < x_2 < a\}.$$

Then $\gamma = \{x_1 = 0,\ 0 < x_2 < a\}$. Moreover, as concerns the triangulations of Ω_1 and Ω_2, we take the one illustrated in Fig. 3. The reason that such triangulations are chosen is already described above. We note that in this case all assumptions of Corollary 2.1 are satisfied. Our test problem is the following. The external force and the boundary condition are taken as $f = (0, f_2)$ and $b = (0,0)$, respectively, where

$$f_2 = 80a^4x_1 + 48x_1^5 + 960a^3x_1x_2 - 960ax_1^3x_2 - 720a^2x_1x_2^2 + 960x_1^3x_2^2 - 480ax_1x_2^3 + 240x_1x_2^4 .$$

Then the exact solution of the target problem is given by

$$\begin{aligned} \tilde{u} &= \left\{ \begin{array}{c} 20(a-x_1)^2(a+x_1)^2x_2(a-x_2)(a-2x_2) \\ -40x_1(a-x_1)(a+x_1)x_2^2(a-x_2)^2 \end{array} \right\} \\ \tilde{p} &= -120a^5x_1 + 80a^3x_1^3 - 24ax_1^5 + 160a^4x_1x_2 - 80a^2x_1^3x_2 \\ &\quad +48x_1^5x_2 + 240a^3x_1x_2^2 - 240ax_1^3x_2^2 - 160a^2x_1x_2^3 + 160x_1^3x_2^3 . \end{aligned}$$

Here we have chosen $\tilde{p}$ subject to $\tilde{p}(0,0) = 0$. The initial guess is taken as

$$\mu^{(0)} = \sin\left(\frac{2\pi}{a}x_2\right)(1,1) .$$

We observe the discrete maximum error:

$$E(k) = \max_{\gamma} |\mu_h^{1,(k)} - \tilde{\mu}^1| + \max_{\gamma} |\mu_h^{2,(k)} - \tilde{\mu}^2|$$

where $\mu_h^{(k)} = (\mu_h^{1,(k)}, \mu_h^{2,(k)})$ is the finite element approximation for $\tilde{\mu} = \tilde{u}|_\gamma = (\tilde{\mu}^1, \tilde{\mu}^2)$ with a mesh parameter $h > 0$. We use the P2/P1 elements.

We chose $a = 0.5$. Convergence histories of $\log E(k)$ are illustrated for several values of θ in Fig. 4, where the line r_b means the convergence history when the convergence factor is equal to 0.86. Figure 4 shows that the claim of Corollary 2.1 is true. Furthermore, we also observe the exponential decay of the error even if θ is greater than 0.142. This result does not contradict Corollary 2.1, since the corollary claims that if $0 < \theta < 0.142$ then the exponential decay of the error occurs.

Since, as described above, we do not analyze the discrete DN iterations directly, the above experiment is no more than an example.

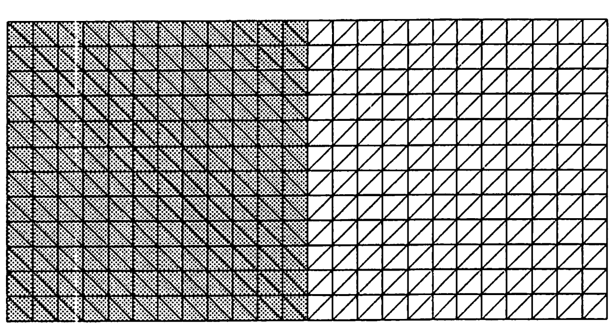

Figure 3. Triangulation of Ω_1 (shaded part) and Ω_2 (adjacent one).

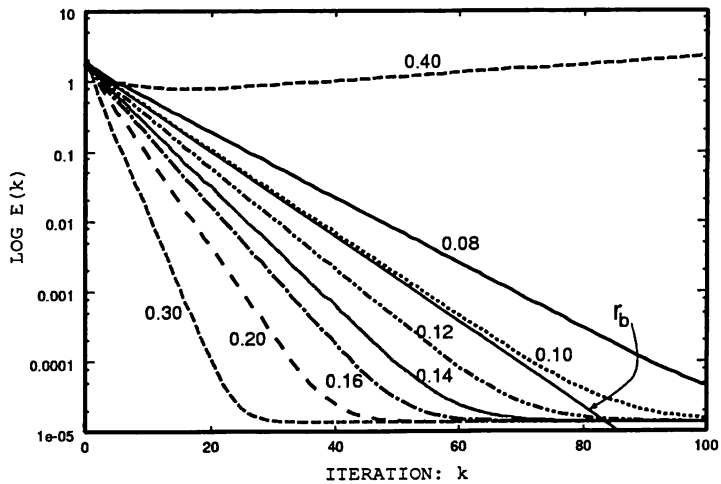

Figure 4. Convergence histories of $\log E(k)$ for several values of θ with $a = 0.5$.

5.2. The inf-sup constant

We show that the convergence factor of the error on γ depends explicitly on the inf-sup constant under Condition (I). As a result, to evaluate the value of the inf-sup constant is of important. Let Ω be a bounded domain in $\mathbb{R}^2$, and let β be the inf-sup constant corresponding to Ω. We assume that Ω is simply connected. Then the equality $\beta^{-1} = \sqrt{\kappa/2}$ holds (Velte [26]), where κ denotes the second Korn's constant (see, for definition, Horgan-Payne [15]). This relation is useful to evaluate the value of β. That is, as concerns the value of κ, we know:

(i) $\kappa \geq 4$ for any bounded domain Ω (Horgan [14]), $\kappa = 4$ holds when Ω is a disk ([15]).

(ii) When Ω is a regular n-side polygon, we have $\kappa \leq 2/(1 - \sin(\pi/n))$ ([15]). In particular, if Ω is a square, we have $\kappa \leq 8 + 4\sqrt{2}$.

5.3. Alternative choice of the artificial boundary condition

In order to fix the idea, we consider the situation given by (3.1) and suppose that $\{w = (w^1, w^2), p\}$ is a solution of the homogeneous Stokes equations (3.4). Then, the function

$$\tau_n[w,p] \equiv \frac{\partial w}{\partial n} - pn = \left[\frac{\partial w^m}{\partial x_j}\right] \cdot n - pn$$

has no obvious physical meaning. There, of course, as long as we regard w and p as elements in $H^1(\Omega)$ and $L^2(\Omega)$, respectively, $\tau_n[w,p]$ is understood as a functional over V which is defined by

$$\langle T, \eta \rangle = a(w, \psi) - (p, \operatorname{div} \psi)_{L^2(\Omega)} \qquad \forall \eta \in V$$

where $\psi \in K^1(\Omega)$ is an extension of η into Ω. As a matter of fact, the boundary condition $\tau_n[w,p] = g$ on γ, g being a prescribed function, is the natural boundary condition corresponding to the Dirichlet form $a(u,v)$ in the H^1-theory. From the view point of hydrodynamics, however, we might have to use the *deformation integral form*

$$E(u,v) = \frac{1}{2} \int_\Omega \sum_{1 \leq m,j \leq 2} e_{m,j}(u) e_{m,j}(v) \, dx \quad \text{with } e_{m,j}(u) = \frac{\partial u^m}{\partial x_j} - \frac{\partial u^j}{\partial x_m}$$

instead of $a(u,v)$ as the H^1-ellipticity form. Then, the corresponding natural boundary condition becomes

$$\sigma_n[w,p] \equiv [-p\delta_{m,j} + e_{m,j}(w)] \cdot n = g \quad \text{on } \gamma .$$

Actually, $[-p\delta_{m,j} + e_{m,j}(w)]$ means the stress tensor and $\sigma_n[w,p]$ the normal stress on γ. If we re-formulate our problem and some devices used in this paper with $E(u,v)$ and $\sigma_n[w,p]$, our results remain true. However, the numerical treatment of $\sigma_n[w,p]$ is more complicated than that of $\tau_n[w,p]$. This is the reason that we employ $\tau_n[w,p]$ as the artificial boundary value in the present paper.

5.4. Resolvent Stokes equation

It is worth while to consider a *resolvent Stokes problem*:

$$\begin{cases} \kappa u - \varepsilon\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = b & \text{on } \Gamma \end{cases} \tag{5.1}$$

in which $\kappa \geq 0$ and $\varepsilon > 0$ are prescribed constants. If $\kappa = 0$, then (5.1) is the usual stationary Stokes problem. On the other hand, if $\kappa > 0$, then (5.1) follows from the time discretization of a time-dependent Stokes problem. As concerns the problem of this type, we can obtain similar results (Saito [24]).

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REFERENCES

1. D. N. Arnold, L. R. Scott, and M. Vogelius, Regular inversion of the divergence operator with Dirichlet boundary conditions on a polygon. *Ann. della Scuo. Norm. Super. di Pisa* (1988) **XV**, No. 2, 169–192.
2. I. Babuška, The finite element method with Lagrangian multipliers. *Numer. Math.* (1973) **20**, 179–192.
3. I. Babuška and A. K. Aziz, *Survey lectures on the mathematical foundations of the finite element method. The mathematical foundations of the finite element method with applications to partial differential equations.* Academic Press, 1972.
4. P. Bjørstadt and O. B. Widlund, Iterative methods for the solution of elliptic problems on regions partitioned into substructures. *SIAM J. Numer. Anal.* (1986) **23**, 1097–1120.
5. F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. *R.A.I.R.O.* (1974) **8** (R-2), 129–151.
6. H. Fujita, Remarks on the domain-dependence of convergence rates of a certain DDM. *Lecture at International Conference on Parallel Algorithm for Scientific and Engineering Computations (PASEC)*, May 1995, Chiba, Japan.
7. H. Fujita, Remarks on the domain-dependence of convergence rate of iterations in a certain domain decomposition method – Analysis by the Steklov-Poincaré operator. In: *Collection of Papers on Geometry, Analysis and Mathematical Physics.* World Sci. Publishing, River Edge, NJ, 1997, pp. 71–84.
8. H. Fujita, M. Katsurada, A. Kobari, and Y. Nagasaka, Analytical and numerical study of convergence of the domain decomposition method, I. *Mem. Inst. Sci. Tech. Meiji Univ.* (1996) **35**, No. 8, 103–135.
9. H. Fujita, M. Fukuhara, and N. Saito, On the rate of convergence of iterations in the domain decomposition methods. In: *Proc. of the third China-Japan seminar on numerical mathematics.* Science Press, 1998, pp. 30–43.
10. H. Fujita and N. Saito, An analytical study of optimal speed of convergence of iterations in DDM under certain shape assumptions of domains. In: *Computational Science for the 21st Century.* John Willey & Sons, 1997, pp. 139–148.
11. D. Fujiwara, Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order. *Proc. Japan Acad.* (1967) **43**, 82–86.

12. V. Girault and P.-A. Raviart, *Finite Element Methods of the Navier-Strokes Equations.* Springer-Verlag, 1986.
13. P. Grisvard, *Elliptic Problems in Nonsmooth Domains.* Pitman, 1985.
14. C. O. Horgan, Inequalities of Korn and Friedrichs in elasticity and potential theory. *Z. Angew. Math. Phys.* (1975) **26**, 155–164.
15. C. O. Horgan and L. E. Payne, On inequalities of Korn, Friedrichs and Babuška-Aziz. *Arch. Rational Mech. Anal.* (1983) **82**, No. 2, 165–179.
16. T. Kato, *Perturbation Theory for Linear Operators.* Springer-Verlag, 1978.
17. J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications, I.* Springer-Verlag, 1972.
18. L. D. Marini and A. Quarteroni, A relaxation procedure for domain decomposition methods using finite elements. *Numer. Math.* (1989) **55**, 575–598.
19. A. Quarteroni, Domain decomposition algorithms for the Stokes equations. In: *Proc. of the Eighth International Conference of Domain Decomposition Methods.* John Wiley & Sons, 1996, pp. 431–442.
20. A. Quarteroni and A. Valli, Theory and application of Steklov-Poincaré operators for boundary-value problems: the heterogenous operator case. *Proc. of the Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations.* SIAM, 1991, pp. 58–81.
21. A. Quarteroni and A. Valli, Domain decomposition for the generalized Stokes problem. In: *Proc. of the Third European Conference on Mathematics in Industry.* Kluwer Academic Publishers and B.G. Teubner Stuttgart, 1990, pp. 59–74.
22. A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations.* Oxford University Press, 1999.
23. N. Saito, On the shape of domains and the speed of convergence in a certain DDM for the Stokes equation. In: *GAKUTO International Series, Mathematical Sciences and Applications.* Gakko-tosho Co., Ltd. (to appear).
24. N. Saito, Domain-dependence of convergence rate in some domain decomposition method for a resolvent Stokes equation. In : *Sūrikaisekikenkyūsho Kōkyūroku.* Reseach Institute for Mathematical Sciences, Kyoto University (to appear).
25. N. Saito and H.Fujita, *Remarks on traces of H^1-functions defined in a domain with corners* (to appear).
26. W. Velte, On optimal constants in some inequalities. *Lecture Notes in Mathematics*, Springer-Verlag, 1988, Vol. 1431, pp. 158–168.
27. K. Yosida, *Functional Analysis.* Springer-Verlag, 1980.